Boolean Operations, Joins, and the Extended Low Hierarchy

Lane A. Hemaspaandra [1]
Dept. of Computer Science
University of Rochester
Rochester, NY 14627, USA

Zhigen Jiang [2]
Institute of Software
Chinese Academy of Sciences
Beijing 100080, China

Jörg Rothe [3]
Institut für Informatik
Friedrich-Schiller-Universität Jena
07743 Jena, Germany

Osamu Watanabe [4]
Dept. of Computer Science
Tokyo Institute of Technology
Tokyo 152, Japan

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[2]Supported in part by a postdoctoral fellowship from the Chinese Academy of Sciences, and by grant NSF-CCR-8957604. Work done in part while visiting the University of Rochester and while at McMaster University. Current address: Suite 1100, 123 Front Street, Toronto, Ontario, M5J 2M3 Canada.

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Abstract

We prove that the join of two sets may actually fall into a lower level of the extended low hierarchy [BBS86] than either of the sets. In particular, there exist sets that are not in the second level of the extended low hierarchy, EL_2, yet their join *is* in EL_2. That is, in terms of extended lowness, the join operator can lower complexity. Since in a strong intuitive sense the join does not lower complexity, our result suggests that the extended low hierarchy is unnatural as a complexity measure. We also study the closure properties of EL_2 and prove that EL_2 is not closed under certain Boolean operations. To this end, we establish the first known (and optimal) EL_2 lower bounds for certain notions generalizing Selman's P-selectivity [Sel79], which may be regarded as an interesting result in its own right.

1 Introduction

The low hierarchy [Sch83] provides a yardstick to measure the complexity of sets that are known to be in NP but that are seemingly neither in P nor NP-complete. In order to extend this classification beyond NP, the extended low hierarchy [BBS86] has been introduced (see the surveys [Köb95, Hem93]). An informal way of describing the intuitive nature of these hierarchies might be the following: A set A that is placed in the kth level of the low or the extended low hierarchy contains no more information than the empty set relative to the computation of a Σ_k^p machine (see [MS72, Sto77] for the definition of the Σ levels of the polynomial hierarchy), either because A is so chaotically organized that a Σ_k^p machine is not able to extract useful information from A, or because A is so simple that it has no useful information to offer a Σ_k^p machine.[1] The low and extended low hierarchies have been very thoroughly investigated in many papers (see, e.g., [Sch83, KS85, BBS86, Sch87, Sch89, Ko91, AH92, ABG, Köb94, SL94, HNOS96]). In light of the informal intuition given above—that classifying the level in the extended low hierarchy of a problem or a class gives insight into the amount of polynomial-hierarchy computational power needed to make access to the problem or the class redundant—one main motivation for the study of the extended low hierarchy is to understand which natural complexity classes and problems easily extend the power of the polynomial hierarchy and which do not. Among the important natural classes and problems that have been carefully classified in these terms are the Graph Isomorphism Problem (which in fact is known to be low), bounded probabilistic polynomial time (BPP), approximate polynomial time (APT), the class of complements of sets having Arthur-Merlin games (coAM), the class of sparse and co-sparse sets, the P-selective sets, and the class of sets having polynomial-size circuits (P/poly). Another motivation for the study of the low and extended low hierarchies is to relate their properties to other complexity-theoretic concepts. For instance, Schöning showed that the existence of an NP-complete set (under any "reasonable" reducibility) in the low hierarchy implies a collapse of the polynomial hierarchy [Sch83]. Among the most important recent results about extended lowness

[1] We stress that this is a very loose and informal description. In particular, for the case of the extended low hierarchy, it would be more accurate to say: A set A that is placed in the $(k+1)$st level of the extended low hierarchy, $k > 1$, is such that NP^A contains no more information than $\mathrm{SAT} \oplus L$ relative to the computation of a Σ_k^p machine.

are Sheu and Long's result that the extended low hierarchy is a strictly infinite hierarchy [SL94] and Köbler's optimal location of P/poly in the extended low hierarchy [Köb94]. In this note, we seek to further explore the structure of the extended low hierarchy by studying its interactions with such operations as the join. In particular, we prove properties of EL_2 with regard to its interaction with the join and with Boolean operations. Our results add to the body of evidence that extended lowness does not provide a natural, intuitive measure of complexity.

In light of the many ways in which extended lowness captures certain concepts of low information content (such as all sparse sets and certain reduction closures of the sparse sets—e.g., the Turing closure of the class of sparse sets, which is known to be equal to P/poly) as well as certain concepts of "almost" feasible computation (such as BPP, APT, and P-selectivity, etc.), it might be tempting to assume that extended lowness would provide a reasonable measure of complexity in the sense that a problem's property of being extended low indicates that this problem is of "low" complexity. However, in Section 2, we will prove that *the join operator can lower difficulty as measured in terms of extended lowness*: There exist sets that are not in EL_2, yet their join is in EL_2. Since in a strong intuitive sense the join does not lower complexity, our result suggests that, if one's intuition about complexity is—as is natural—based on reductions, then the extended low hierarchy is not a natural measure of complexity. Rather, it is a measure that is related to the difficulty of information extraction, and it is in flavor quite orthogonal to more traditional notions of complexity. That is, our result sheds light on the orthogonality of "complexity in terms of reductions" versus "difficulty in terms of non-extended-lowness." In fact, our result is possible only since the second level of the extended low hierarchy is not closed under polynomial-time many-one reductions (this non-closure is known, see [AH92], and it also follows as a corollary of our result).

In Section 3, we apply the technique developed in the preceding section to prove that the second level of the extended low hierarchy is not closed under the Boolean operations intersection, union, exclusive-or, and equivalence. Our result will follow from the proof of another result, which establishes the first known (and optimal) EL_2 lower bounds for generalized selectivity-like classes (that generalize Selman's class of P-selective sets [Sel79], denoted P-Sel) such as the polynomial-time membership-comparable sets introduced by Ogihara [Ogi95] and the multi-selective sets introduced by Hemaspaandra et al. [HJRW96]. These results sharply contrast with the known result that all P-selective sets are in EL_2 and they are thus interesting in their own right.

2 Extended Lowness and the Join Operator

The low hierarchy and the extended low hierarchy are defined as follows.

Definition 1 1. [Sch83] For each $k \geq 1$, define $\mathrm{Low}_k \stackrel{\mathrm{df}}{=} \{L \in \mathrm{NP} \mid \Sigma_k^{p,L} = \Sigma_k^p\}$.

2. [BBS86] For each $k \geq 2$, define $\mathrm{EL}_k \stackrel{\mathrm{df}}{=} \{L \mid \Sigma_k^{p,L} = \Sigma_{k-1}^{p,L\oplus\mathrm{SAT}}\}$, where SAT is the set of all satisfiable Boolean formulas.

For sets A and B, their join, $A \oplus B$, is $\{0x \mid x \in A\} \cup \{1x \mid x \in B\}$. Theorem 2 below establishes that the join operator can lower the difficulty measured in terms of extended lowness.

At first glance, this might seem paradoxical. After all, every set that $\leq_m^p$-reduces[2] to a set A or B also reduces to $A \oplus B$, and thus intuition strongly suggests that $A \oplus B$ must be at least as hard as A and B, as most complexity lower bounds (e.g., NP-hardness) are defined in terms of reductions. However, extended lowness merely measures the opacity of a set's internal organization, and thus Theorem 2 is not paradoxical. Rather, Theorem 2 highlights the orthogonality of "complexity in terms of reductions" and "difficulty in terms of non-extended-lowness." Indeed, note Corollary 4, which was first observed by Allender and Hemaspaandra (then Hemachandra) [AH92]. We interpret Theorem 2 as evidence that extended lowness is not an appropriate, natural complexity measure with regard to even very simple operations such as the join.

Theorem 2 There exist sets A and B such that $A \notin \mathrm{EL}_2$ and $B \notin \mathrm{EL}_2$, and yet $A \oplus B \in \mathrm{EL}_2$.

Lemma 3 below will be used in the upcoming proof of Theorem 2. First, we fix some notations. Fix the alphabet $\Sigma = \{0, 1\}$. Let Σ^* denote the set of all strings over Σ. For each set $L \subseteq \Sigma^*$, $L^{=n}$ ($L^{\leq n}$) is the set of all strings in L having length n (less than or equal to n), and $\|L\|$ denotes the cardinality of L. Let Σ^n be a shorthand for $(\Sigma^*)^{=n}$. Let $\leq_{\mathrm{lex}}$ denote the standard quasi-lexicographical ordering on Σ^*. The census function of a set L is defined by $\mathit{census}_L(0^n) = \|L^{\leq n}\|$. L is said to be sparse if there is a polynomial p such that for every n, $\mathit{census}_L(0^n) \leq p(n)$. Let SPARSE denote the class of all sparse sets. For each class $\mathcal{C}$ of sets over Σ, define $\mathrm{co}\mathcal{C} \stackrel{\mathrm{df}}{=} \{L \mid \overline{L} \in \mathcal{C}\}$. Let $\mathbb{N}$ denote the set of non-negative integers. To encode a pair of integers, we use a one-one, onto, polynomial-time computable pairing function, $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, that has polynomial-time computable inverses. FP denotes the class of polynomial-time computable functions. We shall use the shorthand NPTM to refer to "nondeterministic polynomial-time Turing machine." For an NPTM M (an NPTM M and a set A, respectively), $L(M)$ ($L(M^A)$) denotes the set of strings accepted by M (relative to A).

Lemma 3 If F is a sparse set and $\mathit{census}_F \in \mathrm{FP}^{F \oplus \mathrm{SAT}}$, then $F \in \mathrm{EL}_2$.

Proof. Let $L \in \mathrm{NP}^{\mathrm{NP}^F}$ via NPTMs N_1 and N_2, i.e., $L = L(N_1^{L(N_2^F)})$. Let $q(n)$ be a polynomial bounding the length of all queries that can be asked in the run of $N_1^{L(N_2^F)}$ on inputs of length n. Below we describe an NPTM N with oracle $F \oplus \mathrm{SAT}$:

On input x, $|x| = n$, N first computes $\mathit{census}_F(0^i)$ for each relevant length $i \leq q(n)$, and then guesses all sparse sets up to length $q(n)$. Knowing the exact census of F, N can use the F part of its oracle to verify whether the guess for $F^{\leq q(n)}$ is correct, and rejects on all incorrect paths. On the correct path, N uses itself, the SAT part of its oracle, and the correctly guessed set $F^{\leq q(n)}$ to simulate the computation of $N_1^{L(N_2^F)}$ on input x.

Clearly, $L(N^{F \oplus \mathrm{SAT}}) = L$. Thus, $\mathrm{NP}^{\mathrm{NP}^F} \subseteq \mathrm{NP}^{F \oplus \mathrm{SAT}}$, i.e., $F \in \mathrm{EL}_2$. □

Proof of Theorem 2. $A \stackrel{\mathrm{df}}{=} \bigcup_{i \geq 0} A_i$ and $B \stackrel{\mathrm{df}}{=} \bigcup_{i \geq 0} B_i$ are constructed in stages. In order to show $A \notin \mathrm{EL}_2$ and $B \notin \mathrm{EL}_2$ it suffices to ensure in the construction that $\mathrm{NP}^A \not\subseteq \mathrm{coNP}^{A \oplus \mathrm{SAT}}$ and $\mathrm{NP}^B \not\subseteq \mathrm{coNP}^{B \oplus \mathrm{SAT}}$ (and thus, $\mathrm{NP}^{\mathrm{NP}^A} \not\subseteq \mathrm{NP}^{A \oplus \mathrm{SAT}}$ and $\mathrm{NP}^{\mathrm{NP}^B} \not\subseteq \mathrm{NP}^{B \oplus \mathrm{SAT}}$).

[2] For sets X and Y, $X \leq_m^p Y$ if and only if there is a polynomial-time computable function f such that $X = \{x \mid f(x) \in Y\}$.

Define function t inductively by $t(0) \stackrel{\text{df}}{=} 2$ and $t(i) \stackrel{\text{df}}{=} 2^{2^{2^{t(i-1)}}}$ for $i \geq 1$. Let $\{N_i\}_{i\geq 1}$ be a fixed enumeration of all coNP oracle machines having the property that the runtime of each N_i is independent of the oracle and each machine appears infinitely often in the enumeration. Define

$$L_A \stackrel{\text{df}}{=} \{0^{t(i)} \mid (\exists j \geq 1)\,[i = \langle 0, j\rangle \,\wedge\, \|A \cap \Sigma^{t(i)}\| \geq 1]\},$$

$$L_B \stackrel{\text{df}}{=} \{0^{t(i)} \mid (\exists j \geq 1)\,[i = \langle 1, j\rangle \,\wedge\, \|B \cap \Sigma^{t(i)}\| \geq 1]\}.$$

Clearly, $L_A \in \mathrm{NP}^A$ and $L_B \in \mathrm{NP}^B$. In stage i of the construction, at most one string of length $t(i)$ will be added to A and at most one string of length $t(i)$ will be added to B in order

(1) to ensure $L(N_j^{A_i \oplus \mathrm{SAT}}) \neq L_A$ if $i = \langle 0, j\rangle$ (or $L(N_j^{B_i \oplus \mathrm{SAT}}) \neq L_B$, respectively, if $i = \langle 1, j\rangle$), and

(2) to encode an easy to find string into A if $i = \langle 1, j\rangle$ (or into B if $i = \langle 0, j\rangle$) indicating whether or not some string has been added to B (or to A) in (1).

Let A_{i-1} and B_{i-1} be the content of A and B prior to stage i. Initially, let $A_0 = B_0 = \emptyset$. Stage i is as follows:

First assume $i = \langle 0, j\rangle$ for some $j \geq 1$. If it is the case that no path of $N_j^{A_{i-1}\oplus \mathrm{SAT}}(0^{t(i)})$ can query all strings in $\Sigma^{t(i)} - \{0^{t(i)}\}$ and $N_j^{A_{i-1}\oplus \mathrm{SAT}}(0^{t(i)})$ cannot query any string of length $t(i+1)$ (otherwise, just skip this stage—we will argue later that the diagonalization still works properly), then simulate $N_j^{A_{i-1}\oplus \mathrm{SAT}}$ on input $0^{t(i)}$. If it rejects (in the sense of coNP, i.e., if it has one or more rejecting computation paths), then fix some rejecting path and let w_i be the smallest string in $\Sigma^{t(i)} - \{0^{t(i)}\}$ that is not queried along this path, and set $A_i := A_{i-1} \cup \{w_i\}$ and $B_i := B_{i-1} \cup \{0^{t(i)}\}$. Otherwise (i.e., if $0^{t(i)} \in L(N_j^{A_{i-1}\oplus \mathrm{SAT}})$), set $A_i := A_{i-1}$ and $B_i := B_{i-1}$. The case of $i = \langle 1, j\rangle$ is analogous: just exchange A and B. This completes the construction of stage i.

Since each machine N_i appears infinitely often in our enumeration and as the $t(i)$ are strictly increasing, it is clear that for only a finite number of the $N_{i_1}, N_{i_2}, \ldots$ that are the same machine as N_i can it happen that stage i_k must be skipped (in order to ensure that w_{i_k}, if needed to diagonalize against N_{i_k}, indeed exists, or that the construction stages do not interfere with each other), and thus each machine N_i is diagonalized against eventually. This proves that $A \notin \mathrm{EL}_2$ and $B \notin \mathrm{EL}_2$. Now observe that $A \oplus B$ is sparse and that *census*$_{A\oplus B} \in \mathrm{FP}^{A\oplus B}$. Indeed,

$$\textit{census}_{A\oplus B}(0^n) = 2(\|A \cap \{0, 00, \ldots, 0^{n-1}\}\| + \|B \cap \{0, 00, \ldots, 0^{n-1}\}\|).$$

Thus, by Lemma 3, $A \oplus B \in \mathrm{EL}_2$. □

Corollary 4 [AH92] EL_2 is not closed under $\leq_m^p$-reductions.

In contrast to the extended low hierarchy, every level of the low hierarchy within NP is clearly closed under $\leq_m^p$-reductions. Thus, the low hierarchy analog of Theorem 2 cannot hold.

Fact 5 $(\forall k \geq 0)\,(\forall A, B)\,[(A \notin \mathrm{Low}_k \,\vee\, B \notin \mathrm{Low}_k) \implies A \oplus B \notin \mathrm{Low}_k]$.

Proof. Assume $A \oplus B \in \mathrm{Low}_k$. Since for all sets A and B, $A \leq_m^p A \oplus B$ and $B \leq_m^p A \oplus B$, the closure of Low_k under $\leq_m^p$-reductions implies that both A and B are in Low_k. $\Box$

One of the most interesting open questions related to the results presented in this note is whether the join operator also can *raise* the difficulty measured in terms of extended lowness. That is, do there exist sets A and B such that $A \in \mathrm{EL}_k$ and $B \in \mathrm{EL}_k$, and yet $A \oplus B \notin \mathrm{EL}_k$ for, e.g., $k = 2$? Or is the second level of the extended low hierarchy (and more generally, are *all* levels of this hierarchy) closed under join? Regarding potential generalizations of our result, we conjecture that Theorem 2 can be generalized to higher levels of the extended low hierarchy. Such a result, to be sure, would probably require some new technique such as a clever modification of the lower-bound technique for constant-depth Boolean circuits developed by Yao, Håstad, and Ko (see, e.g., [Hås89, Ko91]).

3 EL_2 is not Closed Under Certain Boolean Connectives

In this section, we will prove that the second level of the extended low hierarchy is *not* closed under the Boolean connectives union, intersection, exclusive-or, or equivalence. We will do so by combining the technique of the previous section with standard techniques of constructing P-selective sets. To this end, we first seek to improve the known EL_2 lower bounds of P/poly, the well-studied class of sets having polynomial-size circuits [KL80]. To wit, we will show that certain generalizations of the class of P-selective sets, though still contained in P/poly [Ogi95, HJRW96], are not contained in EL_2. As interesting as this result may be in its own right, its proof will even provide us with the means required to show the above-mentioned main result of this section: EL_2 is not closed under certain Boolean connectives (and indeed P-selective sets can be used to witness the non-closure). This extends the main result of Hemaspaandra and Jiang [HJ95], namely that P-Sel is not closed under those Boolean connectives.

Let us first recall the following generalizations of Selman's P-selectivity. Ogihara introduced the P-membership comparable sets [Ogi95] and the present paper's authors ([HJRW96], see also [Rot95]) introduced the notion of multi-selectivity as defined in Definition 7.

Definition 6 [Ogi95] Fix a positive integer k. A function f is called a k-membership comparing function for a set A if and only if for every $w_1, \ldots, w_m$ with $m \geq k$,

$$f(w_1, \ldots, w_m) \in \{0,1\}^m \quad \text{and} \quad (\chi_A(w_1), \ldots, \chi_A(w_m)) \neq f(w_1, \ldots, w_m),$$

where χ_A denotes the characteristic function of A. If in addition $f \in \mathrm{FP}$, A is said to be polynomial-time k-membership comparable. Let P-mc(k) denote the class of all polynomial-time k-membership comparable sets.

We can equivalently (i.e., without changing the class) require in the definition that $f(w_1, \ldots, w_m) \neq (\chi_A(w_1), \ldots, \chi_A(w_m))$ must hold only if the inputs $w_1, \ldots, w_m$ happen to be *distinct*. This is true because if there are r and t with $r \neq t$ and $w_r = w_t$, then f simply outputs a length m string having a "0" at position r and a "1" at position t.

Definition 7 Fix a positive integer k. Given a set A, a function $f \in$ FP is said to be an $\mathrm{S}(k)$-selector for A if and only if f satisfies the following property: For each set of distinct input strings $y_1, \ldots, y_n$,

1. $f(y_1, \ldots, y_n) \in \{y_1, \ldots, y_n\}$, and
2. if $\|A \cap \{y_1, \ldots, y_n\}\| \geq k$, then $f(y_1, \ldots, y_n) \in A$.

The class of sets having an $\mathrm{S}(k)$-selector is denoted by $\mathrm{S}(k)$.

It is easy to see that $\text{P-mc}(1) = \text{P}$ and $\mathrm{S}(1) = \text{P-Sel}$. Furthermore, though the hierarchies $\bigcup_k \text{P-mc}(k)$ and $\bigcup_k \mathrm{S}(k)$ are properly infinite, they both are still contained in P/poly [Ogi95, HJRW96]. Among a number of other results, all the relations between the classes $\text{P-mc}(j)$ and $\mathrm{S}(k)$ are completely established in Hemaspaandra et al. [HJRW96]. These relations are stated in Lemma 8 below, as they'll be referred to in the upcoming proof of Theorem 9.

Lemma 8 [HJRW96]

1. $\text{P-mc}(2) \not\subseteq \bigcup_{k \geq 1} \mathrm{S}(k)$.
2. For each $k \geq 1$, $\mathrm{S}(k) \subset \text{P-mc}(k+1)$ and $\mathrm{S}(k) \not\subseteq \text{P-mc}(k)$.[3]

The following result establishes a structural difference between Selman's P-selectivity and the generalized selectivity introduced above: Though clearly $\mathrm{S}(1) = \text{P-Sel} \subseteq \mathrm{EL}_2$ [ABG] and $\text{P-mc}(1) = \text{P} \subseteq \mathrm{EL}_2$, we show that there are sets (indeed, sparse sets) in $\mathrm{S}(2) \cap \text{P-mc}(2)$ that are not in EL_2. Previously, Allender and Hemaspaandra [AH92] have shown that P/poly (and indeed SPARSE and coSPARSE) is not contained in EL_2. Theorem 9 and Corollary 10, however, extend this result and give the first known (and optimal) EL_2 lower bounds for generalized selectivity-like classes.

Theorem 9 $\quad \text{SPARSE} \cap \mathrm{S}(2) \cap \text{P-mc}(2) \not\subseteq \mathrm{EL}_2$.

Proof. Let t be the function defined in the proof of Theorem 2 that gives triple-exponentially spaced gaps. Let $T_k \stackrel{\text{df}}{=} \Sigma^{t(k)}$, for $k \geq 0$, and $T \stackrel{\text{df}}{=} \bigcup_{k \geq 0} T_k$. Let EE be defined as $\bigcup_{c \geq 0} \mathrm{DTIME}[2^{c2^n}]$. We will construct a set B such that

(a) $B \subseteq T$,

(b) $B \in \mathrm{EE}$,

(c) $\|B \cap T_k\| \leq 1$ for each $k \geq 0$, and

(d) $B \notin \mathrm{EL}_2$.

[3] This generalizes to k larger than 1 a result of Ogihara who proves that the P-selective sets are strictly contained in $\text{P-mc}(2)$ [Ogi95] as well as the known fact that P-Sel is strictly larger than P [Sel79].

Note that it follows from (a), (b), and (c) that B is a sparse set in $\mathrm{S}(2)$. Indeed, any input to the $\mathrm{S}(2)$-selector for B that is not in T (which can easily be checked) is not in B by (a) and may thus be ignored. If all inputs that are in T are in the same T_k then, by (c), the $\mathrm{S}(2)$-promise (that B contains at least two of the inputs) is never satisfied, and the selector may thus output an arbitrary input. On the other hand, if the inputs that are in T fall in more than one T_k, then for all inputs of length smaller than the maximum length, it can be decided by brute force whether or not they belong to B—this is possible, as $B \in \mathrm{EE}$ and the T_k are triple-exponentially spaced. From these comments, the action of the $\mathrm{S}(2)$-selector is clear.

By Lemma 8, B is thus in P-mc(k) for each $k \geq 3$. But since $\mathrm{S}(2)$ and P-mc(2) are incomparable (again by Lemma 8), we still must argue that $B \in$ P-mc(2). Again, this follows from (a), (b), and (c), since for any fixed two inputs, u and v, if they are of different lengths, then the smaller one can be solved by brute force; and if they have the same length, then it is impossible by (c) that $(\chi_B(u), \chi_B(v)) = (1,1)$. In each case, one out of the four possibilities for the membership of u and v in B can be excluded in polynomial time. Hence, $B \in$ P-mc(2).

For proving (d), we will construct B such that $\mathrm{NP}^B \not\subseteq \mathrm{coNP}^{B\oplus\mathrm{SAT}}$ (which clearly implies that $\mathrm{NP}^{\mathrm{NP}^B} \not\subseteq \mathrm{NP}^{B\oplus\mathrm{SAT}}$). Define

$$L_B \stackrel{\mathrm{df}}{=} \{0^n \mid (\exists x)\,[|x| = n \,\wedge\, x \in B]\}.$$

Clearly, $L_B \in \mathrm{NP}^B$. As in the proof of Theorem 2, let $\{N_i\}_{i\geq 1}$ be a standard enumeration of all coNP oracle machines satisfying the condition that the runtime of each N_i is independent of the oracle and each machine is repeated infinitely often in the enumeration. Let p_i be the polynomial bound on the runtime of N_i. The set $B \stackrel{\mathrm{df}}{=} \bigcup_{i\geq 0} B_i$ is constructed in stages. In stage i, at most one string of length n_i will be added to B, and B_{i-1} will have previously been set to the content of B up to stage i. Initially, $B_0 = \emptyset$ and $n_0 = 0$. Stage $i > 0$ is as follows:

Let n_i be the smallest number such that (i) $n_i > n_{i-1}$, (ii) $n_i = t(k)$ for some k, and (iii) $2^{n_i} > p_i(n_i)$. Simulate $N_i^{B_{i-1}\oplus\mathrm{SAT}}(0^{n_i})$.

Case 1: If $N_i^{B_{i-1}\oplus\mathrm{SAT}}(0^{n_i})$ rejects (in the sense of coNP, i.e., if there are one or more rejecting computation paths), then fix some rejecting path and let w_i be the smallest string of length n_i that is not queried along this path. Note that, by our choice of n_i, such a string w_i, if needed, must always exist. Set $B_i := B_{i-1} \cup \{w_i\}$.

Case 2: If $0^{n_i} \in L(N_i^{B_{i-1}\oplus\mathrm{SAT}})$, then set $B_i := B_{i-1}$.

Case 3: If the simulation of $N_i^{B_{i-1}\oplus\mathrm{SAT}}$ on input 0^{n_i} fails to be completed in double exponential (say, $2^{100\cdot 2^{n_i}}$ steps) time (for example, because N_i is huge in size relative to n_i), then abort the simulation and set $B_i := B_{i-1}$.

This completes the construction of stage i.

Since we have chosen an enumeration such that the same machine as N_i appears infinitely often and as the n_i are strictly increasing, it is clear that for only a finite number of the $N_{i_1}, N_{i_2}, \ldots$ that are the same machine as N_i can Case 3 occur (and thus N_i, either directly or via one of its clones,

is diagonalized against eventually). Note that the construction meets requirements (a), (b), and (c) and shows $L_B \neq L(N_i^{B \oplus \mathrm{SAT}})$ for each $i \geq 1$. $\square$

Since EL_2 and $\mathrm{P\text{-}mc}(2)$ are both closed under complementation, we have the following corollary.

Corollary 10 $\mathrm{coSPARSE} \cap \mathrm{coS}(2) \cap \mathrm{P\text{-}mc}(2) \not\subseteq \mathrm{EL}_2$.

When suitably combined with standard techniques of constructing P-selective sets, the proof of the previous theorem even proves that the second level of the extended low hierarchy is not closed under a number of Boolean operations, as we have claimed in the beginning of this section. These results extend the main result of Hemaspaandra and Jiang [HJ95] which says that P-Sel is not closed under those Boolean connectives.

Let us first adopt and slightly generalize some of the formalism used in [HJ95] so as to suffice for our objective. The intuition is that we want to show that certain *widely-spaced* and *complexity-bounded* sets whose definition will be based on the set B constructed in the previous proof are P-selective. Fix some complexity-bounding function f and some wide-spacing function μ such that the spacing is at least as wide as given by the following inductive definition: $\mu(0) = 2$ and $\mu(i+1) = 2^{f(\mu(i))}$ for each $i \geq 0$. Now define for each $k \geq 0$,

$$R_k \stackrel{\mathrm{df}}{=} \{i \mid i \in \mathbb{N} \wedge \mu(k) \leq i < \mu(k+1)\},$$

and the following two classes of languages (where we will implicitly use the standard correspondence between Σ^* and $\mathbb{N}$):

$$\mathcal{C}_1 \stackrel{\mathrm{df}}{=} \{A \subseteq \mathbb{N} \mid (\forall j \geq 0)\, [R_{2j} \cap A = \emptyset \wedge (\forall x, y \in R_{2j+1})\, [(x \leq y \wedge x \in A) \implies y \in A]]\};$$

$$\mathcal{C}_2 \stackrel{\mathrm{df}}{=} \{A \subseteq \mathbb{N} \mid (\forall j \geq 0)\, [R_{2j} \cap A = \emptyset \wedge (\forall x, y \in R_{2j+1})\, [(x \leq y \wedge y \in A) \implies x \in A]]\}.$$

In [HJ95], the following lemma is proven for the particular complexity-bounding function $f'(n) = 2^{\mathcal{O}(n)}$ and for the classes $\mathcal{C}_1'$ and $\mathcal{C}_2'$ having implicit in their definition the particular wide-spacing function that is given by $\mu'(0) = 2$ and $\mu'(i+1) = 2^{2^{(\mu'(i))}}$, $i \geq 0$. However, there is nothing special about these functions f' and μ', i.e., for Lemma 11 to hold it suffices that f and μ relate to each other as required above. In light of this, the proof of Lemma 11 is quite analogous to the proof given in [HJ95].

Lemma 11 $\mathcal{C}_1 \cap \mathrm{DTIME}[f] \subseteq \mathrm{P\text{-}Sel}$ and $\mathcal{C}_2 \cap \mathrm{DTIME}[f] \subseteq \mathrm{P\text{-}Sel}$.

Now we are ready to prove the main result of this section.

Theorem 12 EL_2 is not closed under intersection, union, exclusive-or, or equivalence.

Proof. Using the technique of [HJ95], it is not hard to prove that the set B constructed in the proof of Theorem 9 can be represented as $B = A_1 \cap A_2$ for P-selective sets A_1 and A_2. More precisely, let

$$\begin{aligned} A_1 &\stackrel{\text{df}}{=} \{x \mid (\exists w \in B)\,[|x| = |w| \wedge x \leq_{\text{lex}} w]\}, \\ A_2 &\stackrel{\text{df}}{=} \{x \mid (\exists w \in B)\,[|x| = |w| \wedge w \leq_{\text{lex}} x]\}. \end{aligned}$$

Since $B \in \mathrm{EE}$ and is triple-exponentially spaced, we have from Lemma 11 that A_1 and A_2 are in P-Sel and thus in EL_2. On the other hand, we have seen in the previous proof that $B = A_1 \cap A_2$ is not in EL_2. Similarly, if we define

$$\begin{aligned} C_1 &\stackrel{\text{df}}{=} \{x \mid (\exists w \in B)\,[|x| = |w| \wedge x <_{\text{lex}} w]\}, \\ C_2 &\stackrel{\text{df}}{=} \{x \mid (\exists w \in B)\,[|x| = |w| \wedge x \leq_{\text{lex}} w]\}, \end{aligned}$$

we have $B = C_1 \,\Delta\, C_2$, where Δ denotes the exclusive-or operation. As before, C_1 and C_2 are in P-Sel and thus in EL_2. Hence, EL_2 is not closed under intersection or exclusive-or. Since EL_2 is closed under complementation, it must also fail to be closed under union and equivalence. □

The proof of the above result also gives the following corollary.

Corollary 13 [HJ95] P-Sel is not closed under intersection, union, exclusive-or, or equivalence.

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